

# Compactifications of F-Theory on Calabi-Yau Threefolds at Constant Coupling

CHANGHYUN AHN <sup>\*</sup> AND SOONKEON NAM <sup>†</sup>

*Department of Physics and  
Research Institute for Basic Sciences,  
Kyung Hee University,  
Seoul 130-701, Korea*

## ABSTRACT

Generalizing the work of Sen, we analyze special points in the moduli space of the compactification of the F-theory on elliptically fibered Calabi-Yau threefolds where the coupling remains constant. These contain points where they can be realized as orbifolds of six torus  $\mathbf{T}^6$  by  $\mathbf{Z}_m \times \mathbf{Z}_n$  ( $m, n = 2, 3, 4, 6$ ). At various types of intersection points of singularities, we find that the enhancement of gauge symmetries arises from the intersection of two kinds of singularities. We also argue that when we take the Hirzebruch surface as a base for the Calabi-Yau threefold, the condition for constant coupling corresponds to the case where the point like instantons coalesce, giving rise to enhanced gauge group of  $Sp(k)$ .

---

<sup>\*</sup>chahn@nms.kyunghee.ac.kr

<sup>†</sup>nam@nms.kyunghee.ac.kr

Our understanding of nonperturbative aspects of  $N = 2$  supersymmetric gauge theories[1] in four dimensions progressed very much. Deeper understanding of the gauge dynamics comes from the study of brane probes in string theory, where there are also a lot of exciting developments in the conjectured string dualities[2]. Among these, the heterotic/type II duality has been studied in detail[3]. In fact, it has been extended to the F-theory/heterotic duality[4] where the heterotic strings compactified on a two torus  $\mathbf{T}^2$  is dual to F-theory in eight dimensions compactified on  $\mathbf{K3}$  which admits an elliptic fibration. F-theory[5] is defined as the compactifications of type IIB string in which the complex coupling changes over the base. The  $\mathbf{K3}$  surface which is a fiber space where the base is one dimensional complex projective space  $\mathbf{CP}^1$  and torus as the fiber is represented by  $y^2 = x^3 + f(z)x + g(z)$  where  $z$  is the coordinate of the base  $\mathbf{CP}^1$  and  $f$  and  $g$  are the polynomials of degree 8, 12 respectively in  $z$ . This describes a torus for each point on the base  $\mathbf{CP}^1$  labelled by the coordinate  $z$ .

Extension of this to the compactification down to six dimensions is interesting for various reasons. First of all, although the string theory answer is rather trivial, the classical field theory on the 3-brane which produces the answer is not so trivial and has been considered in Ref.[6]. Secondly, the orbifold limit of Calabi-Yau threefold(CY3) itself is an interesting object to study. So far the examples which has been considered are the Voisin-Borcea models[7] which are listed in [4] and are the product of a two-torus  $\mathbf{T}^2$  and  $\mathbf{K3}$  divided by a  $\mathbf{Z}_2$  symmetry; they correspond to type IIB compactification with a space-independent coupling constant. It would therefore be interesting to study other examples of constant couplings and the physics of the gauge theories arising from the string theories.

One has the duality[4] between F-theories compactified on CY threefold and heterotic strings on  $\mathbf{K3}$ . For example, the  $SO(32)$  heterotic string compactified on a  $\mathbf{K3}$  surface was discussed in Ref.[8]. One of the interesting results is that one can obtain nonperturbative  $Sp(1)$  extra gauge group when an instanton shrinks to zero size. Furthermore,

when  $k$  instantons collapse at the point in the **K3**, the  $Sp(1)^k$  factor is replaced by the enhanced gauge symmetry of  $Sp(k)$ . These results are also reproduced in Ref.[9] in the context of dual theory of F-theory on an elliptically fibered CY3. On the other hand,  $E_8 \times E_8$  heterotic string compactified on a **K3** has been studied in Ref.[4] where only extra massless tensor multiplets appear as instantons shrink down to zero size. An aspect of generic pointlike instantons for both  $SO(32)$  string and  $E_8 \times E_8$  string has been analyzed in Refs.[9, 10]. Further compactification to four dimensions corresponds to the type II/heterotic string duality considered in Ref.[3].

Sen[11] has shown a precise relation conjectured in Ref.[4] between the F-theory on a smooth elliptic **K3** manifold and a type IIB orientifold on  $\mathbf{T}^2$ . Using the orbifold limit of **K3**, i.e. for the simplest case of  $\mathbf{T}^4/\mathbf{Z}_2$  where the coupling is constant over the base, new insight into the **K3** compactification was obtained. The points of enhanced gauge symmetries in the F-theory corresponds to those of enhanced global  $SO(8)$  symmetries in the Seiberg-Witten gauge theory[1]. It has been found in Ref.[12] further that there exist other points for which the coupling is constant i.e.,  $\tau = i$  or  $\tau = e^{\frac{\pi i}{3}}$ . At these special points, **K3** becomes the orbifolds of four torus  $\mathbf{T}^4/\mathbf{Z}_m$  where  $m = 3, 4, 6$  with the base,  $\mathbf{T}^2/\mathbf{Z}_m$ . At these orbifold points, a singularity analysis shows that exceptional gauge group symmetries appear. Nontrivial superconformal field theories for  $E_{6,7,8}$  type singularities has been discussed[13] in the context of Seiberg-Witten gauge theory. Very recently, generalizing the work of Sen[11], the field theory of 3-brane probes[14] in a compactification of F-theory on a six torus  $\mathbf{T}^6$  by  $\mathbf{Z}_2 \times \mathbf{Z}_2$  [15] with hodge number  $(h^{11}, h^{21}) = (51, 3)$  was considered in Ref.[6]. This has an interpretation in terms of multiple 3-branes probes on an F-theory orientifold as was discussed in Ref.[16].

In this paper, we do the following two things. First, we analyze special points in the moduli space of the compactification of the F-theory on elliptically fibered CY3's where the coupling remains constant, along the lines of Refs.[11, 12]. This is rather straightforward and can be realized as *other* orbifolds of six torus  $\mathbf{T}^6/\mathbf{Z}_m \times \mathbf{Z}_n$  where

$m, n = 2, 3, 4, 6$ . At various types of intersection points between  $G, G' = SO(8), E_{6,7,8}$  types of singularities, we find that the enhancement of gauge symmetries arises from the intersection of two types of singularities, different from the naively expected gauge symmetry of  $G \times G'$ . We find that the naive gauge symmetries get modified due to the interplay of the singularities. In the second part of this paper, we consider the case when the base for the CY3 is a Hirzebruch surface  $F_n$  and realize that the property of the constancy of coupling leads to exactly the coalescence of pointlike instantons for  $SO(32)$  heterotic string.

Let us first consider the compactification of F-theory on elliptically fibered CY3 where the coupling is constant over the base. It has been found in Refs.[4, 17] that the CY3 can be described as an elliptic fibration in the Weierstrass form  $y^2 = x^3 + f(z, w)x + g(z, w)$ .  $z$  and  $w$  are the coordinates on the base  $\mathbf{CP}^1 \times \mathbf{CP}^1$  and  $f$  and  $g$  are the polynomials of degree 8, 12 respectively in each of them. Notice that there exists an exchange symmetry when we exchange the two  $\mathbf{CP}^1$ 's and simultaneously the coefficient of the term,  $z^l w^k$  is exchanged with that of  $z^k w^l$  in each of the terms. The modular parameter  $\tau(z, w)$  of the fiber can be written in terms of the invariant  $j$  function given by

$$j(\tau(z, w)) = \frac{4(24f(z, w))^3}{\Delta(z, w)}, \quad (1)$$

where the discriminant is  $\Delta(z, w) = 4(f(z, w))^3 + 27(g(z, w))^2$ . From now we will consider only the cases in which  $f$  and  $g$  are factorized. That is,  $f(z, w) = \alpha f_1(z)f_2(w)$  and  $g(z, w) = g_1(z)g_2(w)$  where  $\alpha$  is a constant. Note that  $j(\tau(z, w))$  blows up at the zeroes of the discriminant.

The one solution for the case of constant modulus by rescaling  $y$  and  $x$  and setting the overall coefficient to be 1 has been found in Ref.[6]. Thus we get

$$\begin{aligned} f_1(z) &= \prod_{i=1}^4 (z - z_i)^2, & f_2(w) &= \prod_{i=1}^4 (w - w_i)^2, \\ g_1(z) &= \prod_{i=1}^4 (z - z_i)^3, & g_2(w) &= \prod_{i=1}^4 (w - w_i)^3, \end{aligned} \quad (2)$$

where  $z_i$ 's and  $w_i$ 's are constants. This special compactification corresponds to a configuration where the 24 7-branes are grouped into 4 sets of 6 coincident 7-branes located at the points,  $z_1, z_2, z_3, z_4$ . There exists an  $SL(2, \mathbf{Z})$  monodromy around each of fixed points  $z_i$ 's. The same is true at the points  $w = w_i$  because the base is simply a product of the  $\mathbf{CP}^1$ 's. It is obvious that we have  $SO(8)$  singularities at  $z = z_i$  and  $w = w_i$ . The spacetime theory is an  $N = 1$  supersymmetric theory whose field content was found in Refs.[15, 18]. For example, the open string sectors lead to  $SO(8)$  gauge group for each 7-branes coming from two  $\mathbf{Z}_2$  factors for a total enhanced gauge symmetries  $(SO(8))^4 \times (SO(8))^4$ [18].

Now we continue on to carry out the same procedure for other various subspaces of the moduli space on which the elliptic fiber remains constant modulus. As pointed out in Ref.[12], in the limit of  $\alpha \rightarrow 0$ , we get  $j(\tau(z, w)) = 0$  from which  $\tau(z, w) = e^{\frac{\pi i}{3}}$ . The polynomials are given by

$$\begin{aligned} f_1(z) &= 0, \quad g_1(z) = \prod_{i=1}^3 (z - z_i)^4, \\ f_2(w) &= \prod_{i=1}^4 (w - w_i)^2, \quad g_2(w) = \prod_{i=1}^4 (w - w_i)^3, \end{aligned} \quad (3)$$

where the 12(12) zeroes of  $g_1(z)(g_2(w))$  coalesce into 3(4) identical ones of order 4(2) each. In this case, the discriminant,  $\Delta(z, w)$  takes the form of

$$\Delta(z, w) = 27 \prod_{i=1}^3 (z - z_i)^8 \prod_{j=1}^4 (w - w_j)^6. \quad (4)$$

The singularity type from Tate's algorithm[19] at a zero of the discriminant gives rise to the enhancement of gauge symmetries[10]. Each point  $z = z_i$  on the first  $\mathbf{CP}^1$  factor carries a deficit angle of  $\frac{3\pi}{2}$  all three of them together deforming the  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_3$ . For each point  $w = w_i$  on the second  $\mathbf{CP}^1$ , there is a deficit angle of  $\pi$  all four of them deforming  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_2$ . This is related to orientifold of F-theory on  $\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_2$ . In F-theory,  $F_4$  gauge symmetry corresponds to the 'generic'  $E_6$  singularity in the sense that the condition on the polynomial of  $g(z, w)$  splits the double zeroes of it in the  $E_6$

gauge symmetry. Furthermore,  $G_2$  gauge symmetry corresponds to the ‘generic’  $SO(8)$  singularity with different constraint on the polynomial  $g(z, w)$ [10]. Near a zero at  $z_1$  the singular fiber is of  $E_6$  type. On the other hand,  $SO(8)$  type of singularity appears near a zero at  $w_1$ . For simplicity, at the intersection points between these two ‘generic’ singularities the corresponding gauge group is simply the product of  $F_4 \times G_2$ [20]. It is clear that there are no extra enhanced gauge symmetry factor because that one blowup of the base gives rise to  $II$  singularity<sup>‡</sup> on the exceptional divisor[20], leading to non-gauge group. Thus the full enhanced gauge symmetry group is  $(F_4)^3 \times (G_2)^4$  by resolving the singularity for each point of  $z = z_i$  and  $w = w_i$ . From an exchange symmetry between the two  $\mathbf{CP}^1$  factors we have mentioned before we can proceed similarly for the case of compactification F-theory on  $\mathbf{T}^6/\mathbf{Z}_2 \times \mathbf{Z}_3$ .

When the 12 zeroes of  $g_2(w)$  has coalesced into 3 identical ones of order 4 each and those of  $g_1(z)$  are given as before in eq.(3), then we have the following:

$$f_1 = 0, \quad f_2 = 0, \quad g_1(z) = \prod_{i=1}^3 (z - z_i)^4, \quad g_2(w) = \prod_{i=1}^3 (w - w_i)^4, \quad (5)$$

where the discriminant is given by

$$\Delta(z, w) = 27 \prod_{i=1}^3 (z - z_i)^8 \prod_{j=1}^3 (w - w_j)^8. \quad (6)$$

Of course, each point  $w = w_i$  on the second  $\mathbf{CP}^1$  factor has a deficit angle of  $\frac{3\pi}{2}$  all three of them together deforming the  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_3$ . This corresponds to F-theory orientifold on  $\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_3$ . Each singular fiber over the fixed point  $z_1, z_2, z_3$  and  $w_1, w_2, w_3$  is of  $E_6$  type. According to the requirement of the elliptic fibration on the blowup surface having CY, the relation for the blown up surface restricts to the possible resolutions and satisfies CY condition i.e. the sum of coefficients<sup>§</sup> of two intersecting singular types in Kodaira’s list[23] is the coefficient of singular type on the exceptional divisor plus 1[20].

---

<sup>‡</sup> We keep the notations of  $I_n, II, III, IV, \dots$  for the types of fiber in Kodaira’s classification of singularities. (See Refs.[22, 4])

<sup>§</sup> The coefficients  $a_i$  for each type of singularity are listed in Ref.[23]

The first blowup of the base leads to  $IV$  singularity on the exceptional divisor which will produce  $SU(3)$  gauge group. The next blowups in turn appear in the intersection of  $IV \times IV^*$ , known as dual for which the sum of the coefficients of  $IV \times IV^*$  is always 1. So this intersection does not produce the enhancement of gauge group. For the intersection points of two  $E_6$  singularities, for example, at  $z = z_1$  and  $w = w_1$ , we get the gauge group of  $E_6 \times SU(3) \times E_6$  by an extra  $SU(3)$  factor[20]. Therefore the total enhancement of gauge symmetry group is given by  $(E_6)^3 \times (SU(3))^3 \times (E_6)^3$ .

If the 12 zeroes of  $g_2(w)$  has coalesced into 3 zeroes of order 5, 4, 3 each and those of  $g_1(z)$  are the same as before like (5), it is easy to see that we have the following:

$$f_1 = 0, \quad f_2 = 0, \quad g_1(z) = \prod_{i=1}^3 (z - z_i)^4, \quad g_2(w) = (w - w_1)^5 (w - w_2)^4 (w - w_3)^3. \quad (7)$$

Each point  $w = w_i$  on the second  $\mathbf{CP}^1$  factor has a deficit angle of  $\frac{5\pi}{3}, \frac{4\pi}{3}$  and  $\pi$  all together deforming the  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_6$ . We can describe this point as the F-theory orientifold on  $\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_6$ . In this case naive expectation is that the enhanced gauge symmetry group is the product of  $(F_4)^3 \times (E_8 \times E_6 \times G_2)$ . It is sometimes stated that this is not allowed because they violate the CY conditions[20]. However, one may blow up a transversal intersection curves of  $IV^* \times II^*$  fibers *without* violating CY condition[21]. The resolutions for  $IV^* \times II^*$  include  $II, I_0^*, IV, I_0$ . The  $IV$  line cuts the  $I_0^*$  line and  $II$  line. Each of these intersections induces monodromy within the  $IV$  fiber exchanging two of the three rational curves. It turns  $SU(3)$  into  $SU(2)$  type singularity[21]. Therefore,  $(F_4)^3 \times (G_2 \times SU(2) \times SU(3)) \times (E_8 \times E_6 \times G_2)$  gauge symmetry appears. We can do the similar analysis for the gauge group  $(E_8 \times E_6 \times G_2) \times (F_4)^3$  by exchanging  $z$  with  $w$ .

Suppose that we go to the special point where the each point  $z = z_i (w = w_i)$  on the first(second)  $\mathbf{CP}^1$  factor has a deficit angle of  $\frac{5\pi}{3}, \frac{4\pi}{3}$  and  $\pi$  all together deforming the  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_6$  which indicates F-theory on  $\mathbf{T}^6/\mathbf{Z}_6 \times \mathbf{Z}_6$ . The 12 zeroes of  $g_1(z)$  and  $g_2(w)$  coalesce into 3 ones of order 5, 4, 3 each, and  $f_1 = f_2 = 0$ . The naive result for the enhanced gauge symmetry group is  $(E_8 \times E_6 \times G_2)^2$  which violates the CY conditions. In the case of intersection of  $I_0^* \times II^*$  allows us to have the resolution of  $II$  and  $IV$  type

singularities for which the sum of the coefficients of them are less than 1.

Let us consider the case in which for each point  $w = w_i$  on the second  $\mathbf{CP}^1$ , there is a deficit angle of  $\pi$  all four of them deforming  $\mathbf{CP}^1$  to  $\mathbf{T}^2/\mathbf{Z}_2$ , while the first  $\mathbf{CP}^1$  factor remains unchanged. Putting this together, we find

$$\begin{aligned} f_1(z) &= 0, \quad g_1(z) = (z - z_1)^5(z - z_2)^4(z - z_3)^3, \\ f_2(w) &= \prod_{i=1}^4 (w - w_i)^2, \quad g_2(w) = \prod_{i=1}^4 (w - w_i)^3, \end{aligned} \quad (8)$$

where from the type of singularities we get the enhanced gauge symmetry group is  $(E_8 \times E_6 \times G_2) \times (SO(8))^4$  naively which is again not allowed due to the CY conditions by intersecting of  $II^*$  and  $I_0^*$  with the similar argument of the above.

Another possibility is as follows:

$$\begin{aligned} f_1(z) &= (z - z_1)^3(z - z_2)^3(z - z_3)^2, \quad g_1(z) = 0, \\ f_2(w) &= \prod_{i=1}^4 (w - w_i)^2, \quad g_2(w) = \prod_{i=1}^4 (w - w_i)^3, \end{aligned} \quad (9)$$

which corresponds to  $\tau = i$  from  $j(\tau(z, w)) = 13824$ . This time it can be easily checked that the discriminant is given by

$$\Delta(z, w) = 4(z - z_1)^9(z - z_2)^9(z - z_3)^6 \prod_{i=1}^4 (w - w_i)^6. \quad (10)$$

The singular fiber over each fixed points  $z_1, z_2$  is of  $E_7$  type. The other singular fiber over  $z_3$  is of  $SO(8)$  type. At the intersection points near  $z = z_1$  and  $w = w_1$ , the gauge group appears to be  $E_7 \times SU(2) \times SO(8)$  enhanced by an extra  $SU(2)$  factor by the fact that the first blowup for this intersection appears singular type  $III$  on the exceptional divisor using the CY condition again. The intersection of  $III \times III^*$  leads to a  $I_0$  type singularity which does not produce the enhanced gauge symmetry. Hence,  $(E_7 \times E_7 \times SO(7)) \times (SU(2))^2 \times (SO(8))^4$  gauge symmetry appears there.

Finally, we have the case when the 8 zeroes of  $f_1(z)(f_2(w))$  coalesce into 3 ones of order 3, 3 and 2, and  $g_1 = g_2 = 0$  For the intersection of  $III^* \times III^*$ , the first blow



up gives rise to  $I_0^*$  singularity on the exceptional divisor by using the CY condition as discussed above. The argument for the next intersection of  $III^* \times I_0^*$  are given in the previous paragraph. Then the five resolutions correspond to  $I_0, III, I_0^*, III, I_0$  in which there are three possibilities for the type of  $I_0^*$ , i.e.  $SO(8), SO(7)$  or  $G_2$  depending on whether the singularity is split, semi-split or non-split[10]. There are three cases: no factorization in the polynomial  $x^3 + f(z, w)x + g(z, w)$  corresponds to non-split case, a product of three linear factors does split case, a product of linear and quadratic factors does semi-split case. Only semi-split case satisfies the anomaly factorization condition. In this case, the gauge group appears  $(E_7 \times SU(2))^2 \times SO(7)$  at the intersections of two  $E_7$  singularities. Then we get the enhanced gauge symmetry group  $(E_7 \times E_7 \times SO(7))^2 \times (SU(2) \times SO(7) \times SU(2))^4$  with extra  $(SU(2))^4$  factor due to the intersections of  $I_0^* \times III^*$ .

Let us note that among the various gauge groups which can appear, we have the possibility of realizing  $(E_6)^3 \times (E_7 \times E_7 \times SO(7))$ . For this case, the discriminant  $\Delta(z, w)$  vanishes identically since  $f_1(z) = 0$  and  $g_2(w) = 0$ . Thus the corresponding vacuum can not live in the F-theory moduli space where the couplings remain constant we have studied so far but live in the full F-theory moduli space in the sense that the coupling varies. This is also true of the gauge group  $(E_8 \times E_6 \times G_2) \times (E_7 \times E_7 \times SO(7))$ .

We summarize our results in the following table.

model	enhanced gauge group
$\mathbf{T}^6/\mathbf{Z}_2 \times \mathbf{Z}_2$	$(SO(8))^4 \times (SO(8))^4$
$\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_2$	$(F_4)^3 \times (G_2)^4$
$\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_3$	$(E_6)^3 \times (SU(3))^3 \times (E_6)^3$
$\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_6$	$(F_4)^3 \times (G_2 \times SU(2) \times SU(3)) \times (E_8 \times E_6 \times G_2)$
$\mathbf{T}^6/\mathbf{Z}_4 \times \mathbf{Z}_2$	$(E_7 \times E_7 \times SO(7)) \times (SU(2))^2 \times (SO(8))^4$
$\mathbf{T}^6/\mathbf{Z}_4 \times \mathbf{Z}_4$	$(E_7 \times E_7 \times SO(7)) \times (SU(2))^{12} \times (SO(7))^4 \times (E_7 \times E_7 \times SO(7))$

Table 1. Possible enhancements of gauge symmetry for various F-theory orbifolds

We can compare our findings with those in the table 8 of Ref[20]. In the above table, we restricted to only the cases of intersections between  $I_0^*, II^*, III^*$  and  $IV^*$  singularity types. The above five rows correspond to exactly  $I_0^* \times I_0^*, IV^* \times I_0^*, IV^* \times IV^*, IV^* \times II^*, III^* \times I_0^*$  and  $III^* \times III^*$  respectively when we intersect the specific two zeroes  $z = z_1$  and  $w = w_1$ . Our  $j(\tau(z, w))$  is related to their  $J$  up to constant.

In the remainder of paper, we would like to consider the  $SO(32)$  heterotic string. Consider the following elliptic fibration over Hirzebruch surface  $F_n$  as a base for the CY3 with  $z$  the coordinate of  $\mathbf{CP}^1$  fiber of  $F_n$  and  $w$  the coordinate on the base. This Weirstrass form may be put into the more restrictive form[9, 22] as follows:

$$y^2 = x^3 + f(z, w)x + g(z, w) = (x - \beta(z, w)) (x^2 + \beta(z, w)x + \gamma(z, w)), \quad (11)$$

which gives a section along  $x = \beta(z, w)$ ,  $y = 0$ . The functions  $\beta$  and  $\gamma$  can be represented in a sufficiently generic form as follows[9, 22]

$$\beta(z, w) = Bz^4 + Cz, \quad \gamma(z, w) = Az^8 - 4BCz^5 - 2C^2z^2. \quad (12)$$

Since we constrain above to be a CY space,  $A$  is a polynomial of degree  $8 + 4n$  in  $w$ ,  $B$  is of degree  $4 + 2n$ , and  $C$  is of degree  $4 - n$ .

We consider the nonzero constant modulus case. Using the known relation of the modular parameter in terms of the  $j$  function, we get the following relation in order that  $j(\tau(z, w))$  should be independent of  $z$  and  $w$  for nonzero  $g(z, w)$ ,

$$\frac{f^3}{g^2} = \frac{(\gamma - \beta^2)^3}{(\beta\gamma)^2} = -\frac{27}{4}. \quad (13)$$

Notice that unlike the case of the compactification to 8 dimensions, where the ratio of  $f^3$  and  $g^2$  was an arbitrary nonzero constant, here the value of the ratio gets fixed. For this case actually the discriminant has to vanish. This means that we must have  $A = -2B^2$  as we can see from the factorized form of the discriminant

$$\Delta(z, w) = 4f(z, w)^3 + 27g(z, w)^2 = z^{18}(A + 2B^2)^2((4A - B^2)z^6 - 18BCz^3 - 9C^2). \quad (14)$$

For this special case, we get constant couplings. Now let us be more specific. Suppose  $A + 2B^2$  is of order  $k$  in  $w$ . Then, we can put

$$A + 2B^2 = \prod_{i=1}^k (w - w_i). \quad (15)$$

In general the loci of the singularities  $w_i$  can be different. The zero of the discriminant is of order  $2k$  in  $w$  for arbitrary value of  $z$ . According to Ref.[19], it is clear that we have  $I_{2k}$  fiber type. The gauge group related with  $I_{2k}$  is  $SU(2k)$ . We also observe that this case has a remarkable coincidence with the case where the point like instantons coalesce[8], where the enhanced gauge group is  $Sp(k)$ .

There are other special points when  $f$  or  $g$  vanishes, rendering the modular parameter of the fiber to be a constant. First, when  $f = 0$ , i.e.  $\beta^2 = \gamma$ , we get  $j(\tau(z, w)) = 0$ , and  $\tau = e^{\frac{\pi i}{3}}$ . Here we must have  $C = 0$  and  $A = B^2$  for generic value of  $z$ . Then  $g = -\prod_{i=1}^{4+2n} (w - w_i)^3 z^{12}$ , and if none of the  $w_i$ 's coincide, we get  $SO(8)$  type singularity of the discriminant and thus an enhanced gauge symmetry of  $(G_2)^{4+2n}$  for 'generic' singularity! Some of these might be related to the orbifold cases we discussed in the first

part of the paper. In the case of the orbifold  $\mathbf{T}^6/\mathbf{Z}_3 \times \mathbf{Z}_2$  as done in equation (3) we found that  $g(z, w) = \prod_{i=1}^3 (z - z_i)^4 \prod_{j=1}^4 (w - w_j)^3$ . Merging the zeroes at  $z_i$ 's will produce exactly the above result for  $n = 0$  case. On the other hand, for the case of  $\beta\gamma = 0$  where  $j(\tau(z, w)) = 13824$ , the following relations  $B = C = 0$  or  $A = C = 0$  hold, we get  $\Delta \sim \prod_{i=1}^{4+2n} (w - w_i)^6 z^{24}$ , hence we expect again an enormous gauge enhancement of  $(G_2)^{4+2n}$ . Finally when we consider the case of  $A = C = 0$ , we have  $f = -\prod_{i=1}^{4+2n} (w - w_i)^2 z^8$  which can be realized as the orbifold  $\mathbf{T}^6/\mathbf{Z}_4 \times \mathbf{Z}_2$  with  $n = 0$  by coalescing  $z_i$ 's to vanish in Eq.(9).

Recently, the enhanced gauge group in four dimensions by  $SU(n)$  singular fibers has been studied[24] in the context of F-theory compactifications on CY4. Certain classes of gauge symmetry enhancement have been worked out but certainly more works have to be done especially the cases of colliding singularities. It would be interesting to extend our analysis for these cases. One might also consider the compactification of F-theory on a large class of CY4's of the form  $(\mathbf{K3} \times \mathbf{K3})/\mathbf{Z}_2$  in four dimensions. The properties of these CY4 are even less known, but for the elliptically fibered cases with CY3 as basis will be the starting point of a future work. For example, the orbifolds of eight torus  $\mathbf{T}^8$  by  $(\mathbf{Z}_2)^3$  limit of this CY4s can be written as an elliptic fibration of the form  $y^2 = x^3 + f(z, w, v)x + g(z, w, v)$  where  $v, w$  and  $z$  are the coordinates on  $(\mathbf{CP}^1)^3$  and  $f$  and  $g$  are polynomials of degrees 8, 12 respectively in all their arguments.

C.A. thanks Jaemo Park for discussions on the subject of orientifold. We thank P.S. Aspinwall for pointing out errors in the original version of the paper. This work is supported in part by Ministry of Education (BSRI-95-2442), by KOSEF (961-0201-001-2) and by CTP/SNU through the SRC program of KOSEF.

## References

- [1] N. Seiberg and E. Witten, Nucl. Phys., **B426** (1994) 19; Nucl. Phys., **B431** (1994) 484.

- [2] J.H. Schwarz, hep-th/9607067; J. Polchinski, hep-th/9607050.
- [3] S. Kachru and C. Vafa, Nucl. Phys., **B450** (1995) 69.
- [4] D.R. Morrison and C. Vafa, Nucl. Phys., **B473** (1996) 74; Nucl. Phys., **B476** (1996) 437.
- [5] C. Vafa, Nucl. Phys., **B469** (1996) 403.
- [6] O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, hep-th/9611222.
- [7] C. Voisin, in: Journées de Géométrie Algébrique d'Orsay( Orsay, 1992), Astérisque No. 218(1993), 273; C. Borcea, in: Essays on Mirror Manifolds vol. 2, International Press (1996) Hong Kong.
- [8] E. Witten, Nucl. Phys., **B460** (1996) 541.
- [9] P.S. Aspinwall and M. Gross, Phys. Lett., **B387** (1996) 735.
- [10] M. Bershadsky et al, hep-th/9605200.
- [11] A. Sen, Nucl. Phys., **B475** (1996) 562.
- [12] K. Dasgupta and S. Mukhi, Phys. Lett., **B385** (1996) 125.
- [13] J.A. Minahan and D. Nemeschansky, hep-th/9608047; hep-th/9610076.
- [14] T. Banks, M.R. Douglas and N. Seiberg, Phys. Lett., **B387** (1996) 278.
- [15] J.D. Blum and A. Zaffaroni, Phys. Lett., **B387** (1996) 71; A. Dabholkar and J. Park, hep-th/9607041.
- [16] M.R. Douglas, D.A. Lowe and J.H. Schwarz, hep-th/9612062.
- [17] E.G. Gimon and C.V. Johnson, Nucl. Phys., **B479** (1996) 285.
- [18] R. Gopakumar and S. Mukhi, hep-th/9607057.
- [19] J. Tate, in Modular Functions of One Variable IV, Lect. Notes in Math. vol. 476, Springer-Verlag (1975) Berlin.
- [20] M. Bershadsky and A. Johansen, hep-th/9610111.
- [21] P.S. Aspinwall, private communications.
- [22] P.S. Aspinwall, hep-th/9611137.
- [23] K. Kodaira, Annals of Math., **77** (1963) 563; Annals of Math., **78** (1963) 1.
- [24] A. Klemm, B. Lian, S-S. Roan and S-T. Yau, hep-th/9701023.